Alternative coupled integrable optical soliton system with higher-order effects

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The system of coupled Hirota equations, which explains the simultaneous propagation of two fields in a nonlinear optical fiber with the inclusion of higher-order linear and self-steepening effects, is considered. By making use of a Painlevé singularity structure analysis, the system is found to be an exactly integrable soliton system for three choices of physical parameters. Two of the soliton conditions are already well studied. For the third system, the soliton solutions are obtained using bilinear forms.

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I. INTRODUCTION

Optical communication through fibers has generated considerable interest in research activities among scientists all over the world. In particular, soliton-type pulse propagation plays a vital role in our modern communication systems [1–3]. This is considered to be the tool of the future in achieving low-loss, cost-effective communication throughout the world. Soliton-type pulse propagation through nonlinear optical fibers is realized by means of an exact counterbalance between the major constraints of the fiber, viz., group velocity dispersion, which broadens the pulse, and self-phase-modulation, which contracts the pulse. Recent experimental achievements have also increased interest in potential applications of optical solitons such as optical switching [3,4]. It is well known that optical bright solitons can be used for long distance communication to drastically increase the bit rate of fiber transmission systems. On the other hand, dark solitons, which are reflectionless radiation modes of the waveguides, also have a localized shape similar to bright solitons, but with complex envelope and nonvanishing asymptotics. In the case of temporal solitons, the group velocity dispersion is known to vanish at a wavelength of 1.3 μm and is positive at larger wavelengths and negative at shorter ones. Since silica optical fibers always have a positive Kerr coefficient, the two different signs of the group velocity dispersion support two different types of solitons, dark in the former case and bright in the latter case [1–4].

With the current interest in using solitons as pulse bits in long optical fibers for communication purposes, it is important for us to reevaluate the practicality of using analytical techniques for predicting the behavior of such bits with suitable linear and nonlinear optical effects. Since such pulses are near a pure soliton solution, it becomes feasible to use analytical soliton techniques and the potential for obtaining useful analytical results becomes very high. A slowly varying amplitude electromagnetic wave in a nonlinear medium is usually described by the nonlinear Schrödinger (NLS) equation [5]. In order to increase the bit rate, it is necessary to decrease the pulse width. As the pulse lengths become comparable to the wavelength, however, the NLS equation becomes inadequate, additional terms have to be included, and the resulting pulse propagation is called a higher-order nonlinear Schrödinger equation [1]. This equation includes effects like third-order dispersion (TOD), self-steepening (SS), and stimulated Raman scattering (SRS).

When solitons consist of several interacting modes, the pulse propagations are described by a system of coupled nonlinear partial differential equations. In the case of wavelength division multiplexing [1–3], we shall consider at least two optical fields simultaneously. Without higher-order effects, the coupled nonlinear Schrödinger (NLS) equations take the form

$$\begin{aligned} &iq_{1z}+c_1q_{1tt}+2(\alpha|q_1|^2+\beta|q_2|^2)q_1=0,\\ &iq_{2z}+c_2q_{2tt}+2(\beta|q_1|^2+\gamma|q_2|^2)q_2=0, \end{aligned} \tag{1}$$

where c_1 and c_2 are group velocity coefficients, α and γ are self-phase-modulation (SPM) coefficients, and β is the cross-phase-modulation (XPM) coefficient. Using the inverse scattering transform method, soliton solutions have been generated when $c_1=c_2=1$ and $\alpha=\beta=\gamma=1$ [6]. But as these equations do not admit exact solitons for arbitrary values of the self- and cross-phase-modulation coefficients, to construct exact soliton solutions, we assume the value of the ellipticity angle to be 35°. With this assumption, it has been shown that the SPM and XPM coefficients are equal and the resulting coupled NLS (CNLS) equation is the well-known completely integrable soliton system also called the Manakov model. The soliton aspects of the Manakov model have been well studied by many authors in different contexts [6–13]. Recently, using Hirota's bilinear form and by introducing additional parameters, the *N*-soliton solutions for the CNLS equations have been constructed and inelastic collisions of solitons have been observed [9]. By constructing additional motion invariant laws based on the concept of degenerative dispersion, in addition to Eq. (1), Zakharov and Schulmann established the integrability of one more system of the form [7]

$$\begin{aligned} &iq_{1z}+q_{1tt}+2(|q_1|^2-|q_2|^2)q_1=0,\\ &iq_{2z}-q_{2tt}+2(-|q_1|^2+|q_2|^2)q_2=0. \end{aligned} \tag{2}$$

Using a Painlevé analysis, the Painlevé properties of Eqs. (1) and (2) have been established [8]. When compared with Eq. (1), Eq. (2) has attracted less attention. In recent years, several completely integrable single and coupled NLS-type equations which admit bright and dark optical solitons have been proposed and well investigated in nonlinear fiber optics [6–20]. Equation (2) can be used to explain the wave propagation in birefringent media in which we have both anomalous and normal dispersion and focusing and defocusing nonlinearity. In addition to the above physical situation, the system is also used to explain two pulses copropagating in optical fibers.

If we are propagating high-intensity ultrashort pulses through optical glass fiber, then the Manakov model is found to be inadequate and one has to incorporate higher-order linear and nonlinear effects. In order to increase the transmission capacity of the network systems, one has to increase the number of channels with minimum frequency difference. But, in reality, even a single-mode fiber admits a birefringent effect, and hence the two pulses are propagating in orthogonal directions. In this case, depending on the field strength, the field propagating along one direction may change the refractive index of the other one, and vice versa. Excluding the SRS effect, the copropagation of two ultrashort pulses including the effects of TOD and SS is governed by the following generalized coupled equation [11]:

$$\begin{aligned} &iq_{1t}+c_1q_{1zz}+2(\alpha|q_1|^2+\beta|q_2|^2)q_1\\ &\quad -i\varepsilon\{q_{1zzz}+(2\mu_1|q_1|^2+\nu_1|q_2|^2)q_{1z}\\ &\quad +\nu_1q_1q_2^*q_{2z}\}=0, \end{aligned} \tag{3a}$$

$$\begin{aligned} &iq_{2t}+c_2q_{2zz}+2(\beta|q_1|^2+\gamma|q_2|^2)q_2\\ &\quad -i\varepsilon\{q_{2zzz}+(\nu_2|q_1|^2+2\mu_2|q_2|^2)q_{2z}\\ &\quad +\nu_2q_1^*q_2q_{1z}=0. \end{aligned} \tag{3b}$$

The coupled equation proposed by Tasgal and Potasek [11] is the coupled version of the Hirota equation [17], and a coupled higher-order NLS system has been proposed by us [10] and well studied by several authors [11–16].

The generalized version above has been considered here for the purpose of analyzing various possibilities of integrable soliton cases from the point of view of Painlevé analysis. It has already been reported that Eq. (3) admits soliton solutions for the conditions $c_1=c_2,\ \alpha=\beta=\gamma,\ \mu_1=\nu_1=\mu_2=\nu_2=3$ [11]. For the first condition, exact N-soliton solutions have been reported [11,18] which correspond to bright solitons. Recently, we have shown from a Painlevé analysis that there is one more integrable case, $c_1=c_2=-1,\ \alpha=\beta=\gamma,\ \mu_1=\nu_1=\mu_2=\nu_2=-3$, corresponding to a dark-dark soliton pair, which has not been analyzed for this system [12–14]. In this context, it should be mentioned that for this system, Park and Shin have constructed the Bäcklund transformation and analyzed the dark-dark, bright-dark, and bright-bright pairs of soliton solutions [19,20].

In this paper, the integrability aspects of the coupled Hirota equation are analyzed using a Painlevé singularity structure analysis. The above system is found to be integrable for the following choices of parameters: (i) $c_1=c_2,\ \alpha=\beta=\gamma,\ \mu_1=\nu_1=\mu_2=\nu_2=3$; (ii) $c_1=c_2=-1,\ \alpha=\beta=\gamma,\ \mu_1=\nu_1=\mu_2=\nu_2=-3$; (iii) $c_1=-c_2,\ \alpha=-\beta=\gamma,\ \mu_1=-\mu_2=-\nu_1=\nu_2=3$. As the conditions (i) and (ii) are well studied [11,12], we are interested in analyzing condition (iii). First, in Sec. II, we establish the Painlevé property for condition (iii). Then, in Sec. III, we construct the soliton solutions by a bilinear method.

II. PAINLEVÉ ANALYSIS

Painlevé analysis is a powerful method for identifying the complete integrability properties of nonlinear partial differential equations (NPDEs). Weiss, Tabor, and Carnevale [21] introduced an algorithm for carrying out the Painlevé analysis of given NPDEs.

Under the parametric condition $c_1=-c_2,\ \alpha=-\beta=\gamma$, and $\mu_1=-\mu_2=-\nu_1=\nu_2=3$, Eq. (3) turns out to be

$$\begin{aligned} &iq_{1z}+q_{1tt}+2(|q_1|^2-|q_2|^2)q_1\\ &\quad -i\varepsilon[q_{1ttt}+(6|q_1|^2-3|q_2|^2)q_{1t}-3q_1q_2^*q_{2t}]=0,\\ &iq_{2z}-q_{2tt}-2(|q_1|^2-|q_2|^2)q_2\\ &\quad -i\varepsilon[q_{2ttt}+(3|q_1|^2-6|q_2|^2)q_{2t}+3q_2q_1^*q_{1t}]=0, \end{aligned} \tag{4}$$

where we choose $\alpha=1$ and $c_1=1$. To begin with, Eq. (4) is rewritten in terms of its complex functions by defining $q_1=a,\ q_1^*=b,\ q_2=c,\ q_2^*=d$, and the following equations are obtained:

$$\begin{aligned} &ia_z+a_{tt}+2(ab-cd)a\\ &\quad -i\varepsilon[a_{ttt}+(6ab-3cd)a_t-3adc_t]=0,\\ &-ib_z+b_{tt}+2(ab-cd)b\\ &\quad +i\varepsilon[b_{ttt}+(6ab-3cd)b_t-3bcd_t]=0,\\ &ic_z-c_{tt}-2(ab-cd)c\\ &\quad -i\varepsilon[c_{ttt}+(3ab-6cd)c_t+3bca_t]=0,\\ &-id_z-d_{tt}-2(ab-cd)d\\ &\quad +i\varepsilon[d_{ttt}+(3ab-6cd)d_t+3adb_t]=0. \end{aligned} \tag{5}$$

The Painlevé analysis is carried out by seeking a generalized Laurent series of the form

$$\begin{aligned} a(z,t)&=\phi^p\sum_{j=0}^{\infty} a_j(z,t)\phi^j(z,t),\\ b(z,t)&=\phi^q\sum_{j=0}^{\infty} b_j(z,t)\phi^j(z,t),\\ c(z,t)&=\phi^r\sum_{j=0}^{\infty} c_j(z,t)\phi^j(z,t),\\ d(z,t)&=\phi^s\sum_{j=0}^{\infty} d_j(z,t)\phi^j(z,t) \end{aligned} \tag{6}$$

in the neighborhood of the noncharacteristic movable singular manifold $\phi(z,t)=0$ and searching for conditions under which the solutions are free from movable critical manifolds. The parameters p, q, r, and s are negative integers to be determined.

Assuming the leading order of the solutions to be of the form $a\approx a_0\phi^p$, $d\approx d_0\phi^q$, $c\approx c_0\phi^r$, and $d\approx d_0\phi^s$, they are substituted in Eq. (5) and upon balancing the different terms the following results are obtained:

$$p=q=r=s=-1, \quad a_0b_0-c_0d_0=-\phi_t^2. \tag{7}$$

To find the powers, called resonances, at which the arbitrary functions can enter into the Laurent series, the expressions

$$a=a_0\phi^{-1}+a_j\phi^{j-1}, \quad b=b_0\phi^{-1}+b_j\phi^{j-1},$$

$$c=c_0\phi^{-1}+c_j\phi^{j-1}, \quad d=d_0\phi^{-1}+d_j\phi^{j-1}, \tag{8}$$

are substituted in Eq. (5) and, keeping the leading order terms alone, the following determinant is obtained:

$$\begin{vmatrix} A & -6a_0^2\phi_t & -3(j-2)a_0d_0\phi_t & 6a_0c_0\phi_t \\ -6b_0^2\phi_t & A & 6b_0d_0\phi_t & -3(j-2)b_0c_0\phi_t \\ 3(j-2)b_0c_0\phi_t & -6a_0c_0\phi_t & B & 6c_0^2\phi_t \\ -6b_0d_0\phi_t & 3(j-2)a_0d_0\phi_t & 6d_0^2\phi_t & B \end{vmatrix}=0, \tag{9}$$

where

$$A=(j-1)(j-2)(j-3)\phi_t^3+3(j-2)a_0b_0\phi_t-3(j-2)\phi_t^3$$

and

$$B=(j-1)(j-2)(j-3)\phi_t^3-3(j-2)c_0d_0\phi_t-3(j-2)\phi_t^3.$$

Expanding the determinant and solving by making use of Eq. (7), the resonances are obtained as

$$j=-1,0,0,0,1,2,2,3,4,4,4,5. \tag{10}$$

The resonance at $j=-1$ corresponds to the arbitrariness of the singularity manifold $\phi(z,t)$. From Eq. (7), it is clear that any three of the four functions a_0, b_0, c_0, and d_0 are arbitrary, corresponding to the resonances at $j=0,0,0$. To find the arbitrariness at the other resonance values, to simplify the calculations, Kruskal's reduced manifold ansatz $\phi(z,t)=z+\psi(t)=0$ is applied, and we proceed further by collecting the various powers of ϕ. Collecting the coefficients of $(\phi^{-3},\phi^{-3},\phi^{-3},\phi^{-3})$, the following set of equations is obtained:

$$-2a_0-3i\varepsilon(2a_0b_0a_{0t}-2a_0b_0a_1-2a_0^2b_1-a_{0t}c_0d_0 \\ +a_1c_0d_0-a_0d_0c_{0t}+a_0c_1d_0+2a_0c_0d_1)=0,$$

$$-2b_0-3i\varepsilon(2a_0b_0b_1-2a_0b_0b_{0t}+2b_0^2a_1+b_{0t}c_0d_0 \\ -b_1c_0d_0+b_0c_0d_{0t}-b_0c_0d_1-2b_0c_1d_0)=0,$$

$$2c_0-3i\varepsilon(b_0c_0a_{0t}-a_1b_0c_0+2c_0^2d_1-2c_{0t}c_0d_0-c_1a_0b_0 \\ +a_0b_0c_{0t}-2a_0b_1c_0+2c_0c_1d_0)=0,$$

$$2a_0-3i\varepsilon(2b_0d_0a_1-2a_0b_{0t}d_0-2d_0^2c_1-a_0b_0d_{0t}+a_0b_1d_0 \\ +2c_0d_0d_{0t}+a_0b_1d_1-2c_0d_0d_1)=0. \tag{11}$$

From the above set of equations, the following result is obtained:

$$a_0b_1+a_1b_0=c_0d_1+c_1d_0. \tag{12}$$

Substituting Eq. (12) in Eq. (11), after simplification, it is found that one of the four variables a_1, b_1, c_1, and d_1 is arbitrary, corresponding to the resonance at $j=1$. In a similar manner, collecting the coefficients of $(\phi^{-2},\phi^{-2},\phi^{-2},\phi^{-2})$, $(\phi^{-1},\phi^{-1},\phi^{-1},\phi^{-1})$, $(\phi^0,\phi^0,\phi^0,\phi^0)$, and $(\phi^1,\phi^1,\phi^1,\phi^1)$ one can easily check that the system admits a sufficient number of arbitrary functions at other resonance values. As the system admits a sufficient number of arbitrary functions, it is concluded that the system is expected to be integrable from the Painlevé analysis point of view. As we are not able to construct the Lax pair for Eq. (4), in the following the soliton solutions are generated by the bilinear method.

III. BILINEAR FORM

Once the integrability is proved, the next step is to look for the soliton solutions. Hirota's bilinear approach [17] is used here to obtain soliton solutions. In this, the transformation

$$q_1=\frac{G}{F}, \quad q_2=\frac{H}{F} \tag{13}$$

is introduced, where G and H are complex functions and F is a real function, and the bilinear operator is defined as

$$D_z^m D_t^n G(z,t) F(z',t') = \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n G(z,t) F(z',t') \bigg|_{z=z',t=t'}. \tag{14}$$

Substituting Eq. (13) in Eq. (4), one obtains

$$(iD_z + D_t^2 - i\varepsilon D_t^3) GF = 0, \tag{15a}$$

$$(iD_z - D_t^2 - i\varepsilon D_t^3) HF = 0, \tag{15b}$$

$$D_t^2 FF = 2(|G|^2 - |H|^2). \tag{15c}$$

In order to find the single-soliton solution, the following ansatz is assumed:

$$G = \lambda G_1, \quad H = \lambda H_1, \quad F = 1 + \lambda^2 F_2, \tag{16}$$

where λ is an arbitrary parameter. Substituting Eq. (16) in Eq. (15) and collecting similar powers of λ, the following results are obtained:

$$(iD_z + D_t^2 - i\varepsilon D_t^3) G_1 \times 1 = 0,$$

$$(iD_z - D_t^2 - i\varepsilon D_t^3) H_1 \times 1 = 0 \quad \text{for } \lambda,$$

$$D_t^2 (1 \times F_2 + F_2 \times 1) = 2(|G_1|^2 - |H_1|^2) \quad \text{for } \lambda^2. \tag{17}$$

One can easily check that the solution that is consistent with the system containing Eq. (17) is

$$G_1 = g_1 \exp(\eta_1), \quad H_1 = h_1 \exp(\eta_2),$$

$$F_2 = \frac{(|g_1|^2 - |h_1|^2)}{4\omega_1^2} \exp(\eta_1 + \eta_2), \tag{18}$$

where $\eta_1 = (-\varepsilon\omega_1^3 + i\omega_1^2) z - \omega_1 t$ and $\eta_2 = (-\varepsilon\omega_1^3 - i\omega_1^2) z - \omega_1 t$. Substituting Eq. (18) in Eq. (16) and then in Eq. (13), the one-soliton solutions for the coupled Hirota equations are obtained as

$$q_1 = \frac{\omega_1 g_1 \exp(i\omega_1^2 z)}{\sqrt{|g_1|^2 - |h_1|^2}} \operatorname{sec} h(\varepsilon\omega_1^3 z + \omega_1 t + \eta_0), \tag{19a}$$

$$q_2 = \frac{\omega_1 h_1 \exp(-i\omega_1^2 z)}{\sqrt{|g_1|^2 - |h_1|^2}} \operatorname{sec} h(\varepsilon\omega_1^3 z + \omega_1 t + \eta_0), \tag{19b}$$

where $\eta_0 = \ln(2\omega_1/\sqrt{|g_1|^2 - |h_1|^2})$. From the above, it is clear that the solutions obtained are bright soliton solutions, although we assumed anomalous and normal dispersion in Eq. (4).

In this paper, considering higher-order dispersion and self-steepening, the possibility of soliton-type pulse propagation in a system of coupled Hirota equations is analyzed through Painlevé singularity structure analysis. Using Painlevé analysis, the coupled Hirota system is found to be integrable for the following choices of parameters: (i) $c_1 = c_2$, $\alpha = \beta = \gamma$, $\mu_1 = \nu_1 = \mu_2 = \nu_2 = 3$; (ii) $c_1 = c_2 = -1$, $\alpha = \beta = \gamma$, $\mu_1 = \nu_1 = \mu_2 = \nu_2 = -3$; (iii) $c_1 = -c_2$, $\alpha = -\beta = \gamma$, $\mu_1 = -\mu_2 = -\nu_1 = \nu_2 = 3$. The conditions (i) and (ii) have been well studied in the literature. For the condition (iii), using Painlevé analysis, we proved the existence of a sufficient number of arbitrary functions and hence concluded that the system is integrable. It is also interesting to note that one can check the above condition using the recursion operator method. Using that method, we can show that conditions (i) and (iii) are the next hierarchy of Eqs. (1) and (2). Hence we have derived the hierarchy of the coupled NLS equations. Soliton solutions have been generated using the Hirota bilinearization technique. Although the soliton conditions obtained in this paper are very rigid from the physical point of view, we would like to point out the following. (i) First, from optical soliton theory point of view, our system adds to the already existing integrable soliton theory and can explain the simultaneous propagation of solitons with anomalous and normal dispersion. (ii) In recent years, several experimental groups have investigated the simultaneous propagation of bright and dark solitons in optical fiber. The soliton solutions given in this paper give some idea about the nature of the pulse width, shape, and velocity of the solitons. (iii) As pointed out in Ref. [14], one can also construct solitary wave solutions without imposing any restrictions on the physical parameters so that the results obtained can be related to real experimental situations. In a recent paper, Sakovich and Tsuchida investigated the Painlevé property of symmetrically coupled higher-order NLS equations [22]. Due to symmetry nature of problem, they have not obtained the conditions reported in this paper.

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